Of Induction

from *Essays on the Philosophy of Analysis*
by Charles Babbage
around 1820

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Contents

[Editor's Preface]

[This chapter is from Charles Babbage's *Essays on the Philosophy of Analysis*. The part of the book in which this chapter occurs was called by Babbage in his autobiography "The Philosophy of Invention".

To understand the context of the article *Of Induction*, read two subsections of the appendix of this document: (1) Babbage's notes on this essay from his autobiography and (2) the introduction to *Essays on the Philosophy of Analysis*.

The chapter *Of Induction* appear as sheets 56–67 in Additional MS 37202 from the British Library, totally 24 handwritten pages. The article has been read in and typeset

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Please note that punctuation and capitalization has been added in square brackets. Some repeated words have been removed and paragraphs have been added where determined appropriate. Babbage's original footnotes appear outside square brackets typically as asterisks or daggers. A footnote is added immediately after with the contents of Babbage's footnote.

Babbage included both Latin and French quotes. Translations of these quotes have been obtained, and the translated quotes appear in the text with the original Latin or French quotes in associated footnotes. Two longer quotes appear at the end.

Babbage did not provide section headings. To make it easier to navigate the text, section headings have been added in square brackets.

There are a few words that were illegible in Babbage's manuscript. Question marks have been inserted in square brackets wherever there is an illegible word. In some of these, a likely word has been entered.]

[Introduction]

The term induction when employed in mathematics is not to be understood in precisely the same acceptation as it is used by the followers of Bacon[.] [I]n enquiries of natural philosophy it implies the detection of the physical cause of some phenomena by examining when it is attended with different circumstances[.] [T]hose which are not concerned in inducing the effect are gradually excluded[,] and the efficient cause becomes apparent[.] [T]his however is only the first part of the enquiry[.] [T]he next step consists in determining the laws to which this cause is subjected[,] and[,] by showing the accordance of these laws which are found with the phenomena of nature[,] the correctness of the induction is confirmed and established.

In mathematical enquiries the method of induction is said to be made use of when by examining a few particular cases of a theorem we conclude the truth of some general law[.]

[Binomial Theorem]

[S]uch an instance occurred in investigating the binomial theorem for whole numbers[.] [A] few of the first powers give

1st $1+x$
2nd $1+2x+x^2$
3rd $1+3x+3x^2+x^3$
4th $1+4x+6x^2+4x^3+x^4$
5th $1+5x+10x^2+10x^3+5x^4+x^5$

[T]he first term of all these powers is unity; the coefficient of the second is evidently equal to the exponent of the power to which it is raised[.] [I]f therefore this exponent were n that coefficient would be n[.] [T]he coefficients of the third terms are the triangular numbers which are known to be $\frac{n(n-1)}{2}$[.]

[H]aving observed that the coefficients of the second and third terms are $\frac{n}{1}$ and $\frac{n(n-1)}{1\cdot 2}$ we infer what that of the fourth is $\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$[.] [T]his is itself a process of induction[,] and[,] since it agrees with the numerical value of those terms we have calculated[,] we conclude it true[.] [T]he same inductive process repeated leads us to consider $\frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}$ as the coefficient of the fifth term[.] [H]ence then we conclude that

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{1\cdot2\cdot3}x^3 + \&c.$$

[T]he series on the right hand will always consist of $n+1$ terms[,] and in this respect it agrees with the five cases we have considered[.]

[Remarks on This Process of Reasoning]

[S]o far this formula is a mere induction founded on five particular cases[,] and all its claims to our belief rest on a foundation which[,] if it were a question of natural philosophy[,] might be regarded as extremely narrow[.] [B]ut in cases which refer to algebraical formula[,] our experience of their great regularity dispenses with a multitude of cases.

Before I proceed to explain the second part of which usually accompanies the reasoning just stated and which changes completely the evidence on which it rests from an highly probable to a necessary truth[,] I shall offer a few observations of the process.

[T]he first coefficient being unity[,] we can form no induction from it[.] [I]n fact it does not afford us to make even a reasonable guess what form the second one may take[.] [O]n inspection however that form is evident[,] and we assume it at once to be n[.] [F]rom these two coefficients[,] 1 and n[,] we may make an induction[,] or[,] in other words[,] may form a conjecture[.] [W]e may imagine that the 3rd term will be n^2 and the 4th n^3 &c[.] [T]his satisfies the two first terms[,] but[,] on comparing it with the third[,] it is found to fail[.] [O]n comparing all the third terms which are written[,] it is soon perceived that they are the series of triangular numbers, and consequently of the form $\frac{n(n-1)}{2}$.

[B]y considering the two terms n and $\frac{n(n-1)}{2}$ it is apparent that the latter has a factor more than the former[.] [I]t may therefore be expected that another factor will be added to the next coefficient[,] and[,] since unity was subtracted from the first factor to form the second[,] the same operation repeated may perhaps produce the third[.]

Such appear to be the ideas which would naturally pass through the mind in making the induction we have been considering[.]

[Alternate Argument]

I shall now explain what part of the process which so materially changes our ideas of the nature of the proposition we have arrived at. [I]t is the more important to remark[,] because without putting on the appearance of inductive reasoning[,] it[,] in fact on the foundation of *one single instance*[,] determines the necessary truth of a series of operations extending indefinitely[.]

[T]he step to which I now refer is the following[.] [S]upposing the formula already found[,] we enquire what will be the consequence of multiplying both sides by $1+x$[.] [[1]]

[1] [Babbage writes in the upper-right corner:] [?] [?] [?] [?] and[,] as the first term was not divided[,] the second was divided by 1[,] the third by 2[,] we may imagine the fourth should be divided by three. In this however we find ourselves mistaken[,] and the mistake arise from the circumstance that the three first terms of the denominators coincide with the 3 first terms of the series $0, 1, 2, 3, \&c$ [which is the same as for the series] $0, 1, 1\cdot2, 1\cdot2\cdot3, \ldots$

The operation follows

$$1+\frac{n}{1}x+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{1{\cdot}2{\cdot}3}x^3+[...]\ \ x+\frac{n}{1}x^2+\frac{n(n-1)}{1{\cdot}2{\cdot}3}x^3+[...]$$

[evaluates to]

$$1+\frac{n+1}{1}x+\frac{(n+1)n}{1{\cdot}2}x^2+\frac{(n+1)n(n-1)}{1{\cdot}2{\cdot}3}x^3+$$

[F]rom this it appears that when the formula is multiplied by $1+x$ we have

$$(1+x)^{n+1}=1+\frac{n+1}{1}x+\frac{(n+1)n}{1{\cdot}2}x^2+\frac{(n+1)n(n-1)}{1{\cdot}2{\cdot}3}x^3+[\&c]$$

which is precisely the same result that would have been found had we substituted in it $n+1$ for n. From this we learn that if the formula is true for the nth power it is true for the $(n+1)$th[.] [I]t only remains therefore to show that it is true for some one power[,] the first for example[;] but if $n=1$ it becomes $1+x=x+1$ an identical equation. Consequently it is true for the first power[.]

[N]ow the demonstration we have just gone through proves that if it is true for any one power it is true for the one immediately above[.] [T]herefore it is true for the second; and since for the second, therefore for the third, and so on inductively.

[Newton's Generalised Binomial Theorem]

Having arrived[,] by induction[,] at the binomial theorem when the exponent is a whole number[,] the next step towards perfection was directed by a different guide.

Newton[,] who had observed that the theorem in question is always verified when the exponent is a whole number[,] was led to examine whether it failed in the case of the exponent being a fraction[.] [T]he simplest case of this kind is when $n=1/2$[.] It therefore was the one to which he first applied it[,] and[,] multiplying the expression[,] which this value of n afforded being[,] by itself[,] he found that the product was equal to $1+x$[.] [T]hus he acquired a demonstration for the truth of his formula for the particular value $n=1/2$[.]

[B]ut this was not an induction[.] [I]t rather flowed from a principle of generalisation[,] which will be considered in another section[.] [T]hat transition from $n=1/2$ to $n=1/3$ or $n=1/4$ was very natural and[,] finding that the third and fourth power of the series[,] which resulted from those values of n[,] were equal to $1+x$[,] he concluded that the theorem[,] which now have his name[,] is true for fractions as well as integers.[[2]]

The discovery of this theorem[,] or rather of that part of it which renders it particularly valuable[,] is therefore suggested by a spirit of generalisation[,] and the only proof which Newton had for it[,] rested on induction[.]

[T]his induction might[,] in many cases[,] be converted into certainty by the reasoning explained in a former page[.] [F]or if we have[,] by actual elevation of powers[,] found that it is true when $n=\frac{1}{i}$[,] then it necessarily follows that it will be true whenever $n=\frac{n(i+1)}{i}$.

[2] Newton writes out these three examples and concludes from them in the *Commercium Epistolicum*.

[Euler and Induction]

[Formulas Regarding Prime Numbers]

The method of induction has been employed in the theory of numbers perhaps more frequently than in any other branch of analysis[,] and yet[,] by a singular coincidence[,] there is none in which it so frequently leads to error. The mistake of Fermat in asserting that $2^{2^x} + 1$ is always a prime number is to be here ascribed to this cause[,] as he proposes it as a truth of which he is assured but of which he posses no demonstration.

The number of instances on which his induction was grounded could only be four since Euler has shown that it fails in the fifth number *[3].

In the very paper in which Euler points out the errors into which Fermat had fallen[,] he himself employs the same deliberate instrument and deduced from an induction[,] much better supported than that of Fermat[,] a theorem which has had a great influence in subsequent enquiries relating to the properties of number[.] [H]e observed that *if a and b are neither of them divisible by n then $a^n - b^n$ is always divisible by $n + 1$ if that number is a prime.*

This theorem led him to many similarities of which he remarks.

> "I have fallen upon many other not less elegant theorems in this pursuit, which with that one I think to be required to be valued more, because either in short they are unable to be demonstrated or they may follow from propositions of this kind, which are unable to be demonstrated"[4]

About four years†[5] after[,] a very simple demonstration of this theorem was discovered by Euler which rests on a property of the coefficients of the binomial. In the former paper[,] he had anticipated great difficulty from the circumstance of its being only true when $n + 1$ is a prime number[.] [A] similar reason [was assigned?] by Waring [6] as a cause of the difficulty of proving Wilson's theorem[7.][.] [A]nd although modern analytics have in vain attempted to discover some formula which shall contain only prime numbers, they have been frequently successful in discovering properties of them which are only true of primes.

Few mathematical authors have made so frequent and[,] it may be added[,] so successful use of induction as the illustrious analyst whose writings have furnished us with the example just instanced[.] [8] [T]o those whose researches lead them to investigate

[3]* [Babbage writes] Com. acad Sc. Petrop. Tom. 6 1732 [This is: Leonhard Euler, "Observations Concerning A Certain Theorem Of Fermat And Other Considerations Regarding Prime Numbers", E26, *Commentaries of the St. Petersburg Academy of Sciences*, Vol. 6 (1732/3), 1738, pp. 103–107.]

[4]Commentaries of the St. Petersburgh Academy of Sciences 6 (1732/3), 1738, p. 103-107. Translated by Ian Bruce from "Haec persecutus in multa alia incidi theoremata non minus elegantia, quae eo magis aestimanda esse puto, quod vel demonstrari prorsus nequeant vel ex eiusmodi propositionibus sequantur, quae demonstrari non possunt".

[5]† Leonhard Euler, "The Demonstration Of Certain Theorems Regarding Prime Numbers", *Commentaries of the St. Petersburg Academy of Sciences*, 8, E54, 1741, pp. 141–146. Babbage wrote: "†Com. acad Sc. Petrop. Tom 8 1736"

[6]Edward Waring, 1736–1798

[7]John Wilson (1741–1793). Wilson's theorem states that iff n is prime and over 1, then $(n-1)! \equiv -1 \pmod{n}$

[8]Note at the bottom: *"Thus generally there are hidden truths of this kind, so that the demonstrations of these require both incredible concentration as well as an enormous power of ingenuity." [from Euler's "Example of the use of observation in pure mathematics" E256. The quote is translated by Ian Bruce and Bjørg Tosterud from "Sunt enim plerumque huius generis veritates ita reconditae, ut earum demonstrationes tam incredibilem circumspectionem quam eximiam ingenii vim requirant."] [Translated by the Euler Archive from: "Specimen de usu observationum in mathesi pura - ([?]. [?] 1756 p 187)"]

the secret [string of?] thought[,] they will even present facilities which are rarely offered by other writers[.] [T]his arises in a great measure from the style in which his ideas [[9]] were communicated to the world[,] although possessing to a large extent that spirit of generalisation which has carried this science to so high a point[,] he rarely allows himself to exert its full power but ascends with his readers step by step to conclusions so remote from the simple principles from which he set out that[,] had the beginning and the end of his reasoning only been spread to their attention[,] it might baffle their utmost ingenuity to devine how he had passed the intervening chain.

Euler appears[,] in many of those memoirs[,] to have written down the course of his thoughts just as they presented themselves[,] and he frequently exhibits the trials he had made before he arrived at the successful mode of treating the question[.]

Considerations of this nature will sufficiently apologise for the circumstance that so large a portion of the illustrations I shall offer of the various subjects[,] which will be treated in this volume[,] are selected from the works of Euler. The theory of numbers[,] which abounds in interesting and unexpected relations and which from its peculiar nature seems to demands for its advancement a longer share of original genius than any other branch of mathematics[,] was continually supplying their celebrated author with occasions for calling to his aid the power of inductive reasoning[.] [A]lternately inventing theorems by induction and confirming them by demonstration[,] he appears at different periods to have allowed very different weight to the evidence afforded by the former. In the first of the two memoirs to which I have referred[[10]][,] after stating a theorem he had found by induction[,] he adds, "I do not have a demonstration, yet truly I am most certain concerning the truth of this". [[11]]

[A]fter an interval of four years he thus speaks of the value of inductive reasoning in researches respecting number[:] "But truly I may indicate by several examples that as little as possible should be attributed to inductions in this business" [[12]] and again in the same paper[:] ["]For this reason all numerical properties of this kind, which depend on induction only, I decided long ago to be taken as uncertain, until these either may be fortified by a clearly proved demonstration, or generally they may be refuted." [[13,14]]

[Induction with Complete Certainty]

In the year 1780[,] that is more than forty years after[,] we meet with a paper of the same author which has the following singular title[:] *Being carried out by induction with complete certainty*[[15]][.] Its object is not less curious than its title[,] for it proposes to prove by induction one part of the celebrated theorem of Fermat relative to $n$gonal numbers (that which asserts that every number is the sum of one, two, three, or four squares)[,] a proposition which had been previously placed amongst the number of necessary truths by the complete demonstration which had been given of it both by himself

[9] I allude particularly to his various paper which form parts of Academical collections [this is written on top:] and to [those?] in the Opuscula Analytica

[10] i.e. E26

[11] From E26, translated by Ian Bruce from "cuius quidem demonstrationem quoque non habeo, verum tamen de eius veritate sum certissimus"

[12] From E54, translated by Ian Bruce from "At vero quam parum inductionibus in hoc negotio tribui possit, pluribus exemplis possem declarare"

[13] acta acad. Scient. Imp. Petrop. 1780.

[14] From E54, translated by Ian Bruce from: "Hanc ob rationem omnes huiusmodi numerorum proprietates, quae sola inductione nituntur tam diu pro which incertis habendas esse arbitror, donec illae vel apodicticis demonstrationibus muniantur vel omnino refellantur."

[15] E566, Acta Academiae Scientarum Imperialis Petropolitinae 4, 1784, pp. 38-48. Translated by Ian Bruce and Bjørg Tosterud from "De inductione ad plenam certitudinem evehenda"

and *by* Lagrange[.] [A]t the conclusion of the paper[,] he extends the same reasoning to the case of trigonal numbers. The title of this paper concerns the opinion which its author now entertained of the evidence afforded by this most delicate instrument of discovery[.] [A] single extract will suffice to explain it[:]

> Yet meanwhile these solutions have been carried out by the use of induction to such a degree of certainty, that it may be seen that no further doubt remains. But also perhaps the induction itself may thus be verified by certain reasoning.

[[16]] These *"certain reasoning"*[[17]][,] mentioned in the latter part of the sentence[,] are merely an additional number of conditions which make the truth of the proposition more probable[.] [T]hey by no means partake of demonstrative evidence nor are they in any degree analogous to that part of the proof of the binomial[,] which infers the truth of the nth power from that of the $(n-1)$th[,] and this latter from the $(n-2)$th and so on until the nth is found as depend on the first and this is known to be true by trial.

[Induction for an Expansion]

The whole tenor of this paper[,] whose chief aim remains to be to exhibit the strength of inductive reasoning[,] is strikingly contrasted with that of another which is permitted in [Euler's] *Brief Analytical Works*†[[18]][.] [T]he date of its composition is not given nor are we certainly acquainted with that of the one just mentioned. The object of that contained in the *Brief Analytical Works* is to find the coefficient of x^n in the expansion of $(1+x+x^2)^n$ and to exhibit several curious properties which it proposes. If we from another series having for its coefficients the successive value of this coefficient it will be

$$y + 3y^2 + 7y^3 + 19y^4 + 51y^5 + 141y^6 + \&c.$$

[I]t is now required to find the general term of this series[.] Euler commences his investigations by attempting to find if from induction[.] [T]his leads him to the following value of its *terminus generalis*

$$\frac{1}{10}3^n + \frac{1}{10}(-1)^n + \frac{1}{5}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{3-\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

which he observes coincides with the nine first terms but fails in the tenth. "Therefore this example of illegal induction is the more worthy of note so far, because indeed a case of this kind has not yet happened to me, in which nevertheless the most presentable induction has failed." [[19]]

Having found the inductive process not to be in *this case relied* on[.] [H]e investigates the general term by a direct process and readily finds it to be

[16] E566, pp. 38–9. Translation by Ian Bruce and Bjørg Tosterud from "Interim tamen istae resolutiones per solam inductionem iam ad tanum certitudinis gradum euectae funt, vt nullis amplius dubiis locus relinqui videatur. Quin etiam ipsa inductio fortasse per certas rationes ita corroborari posse videtur, vt instar absolutae demonstrationis spectari possit."

[17] Translated by Ian Bruce and Bjørg Tosterud from "certas rationes"

[18] † [Babbage writes:] Tom 1. p. 48[.] [This is Leonard Euler, "Various methods for inquiring into the innate characters of series" (E551) *Brief analytical works*, vol. 1, Saint Petersburg, 1783, p. 48. [translated by the Euler Archives from "Varia artificia in serierum indolem inquirendi", *Opuscula analytica*]

[19] From "Various methods for inquiring into the innate characters of series" [Translated by the Euler Archive from "Varia Artificia in serierum indolem inquirendi], Opuscula analytica 1, 1783, p. 48–63". Translated by Ian Bruce and Bjørg Tosterud from "Hoc ergo exemplum inductionis illicitae eo magis est notatu dignum, quod mihi quidem eiusmodi casus nondum obtigerit, in quo tam speciosa inductio fefellerit."

$$1+\frac{n(n-1)}{1\cdot 1}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3}+\&c$$

[Conclusions of this Section]

These opinions respecting induction are perhaps not quite so opposed to each other as it might at first be supposed[,] and[,] although they all relate to mere number in which this mode of proof is most likely to fail[,] yet it affords very various degrees of evidence in different cases, which circumstance might sufficiently justify the different terms in which Euler has spoken of it. When the term probable is applied to any truth which is established by such evidence as that we are now considering, it should be observed that it does not apply to the nature of the proposition which is in all cases *necessarily* true or *necessarily* false[,] but it applies to the arguments which induce us to believe one of these to be the case rather than the other[.] [A]ccording to the number and quality of these arguments[,] we decide[,] and it is obvious that they may vary through all degrees from bare probability up to a conviction beyond even moral certainty. To state with precision the reasons which influence our judgement of these degrees would greatly add to the value of this instrument of investigation, but the difficulty of accomplishing this is great[.] [I]t is perhaps increased from the circumstance that we rarely carry out the inductive process without at the same time generalising and also that comparatively speaking it is not so often employed, especially by modern analysis, as the more refined methods of assisting the inventive faculty.

[The Least Figures of Powers of Numbers]

Sensible that in my own case[,] whenever I have been successful in carrying my enquiries a few steps out of the beaten path[,] I have been more indebted to generalisation and to analogical reasoning than to the process of induction[.] I shall not venture to offer many observations on a subject on which my experience is very limited[.]

[Babbage's Example No. 1]

At two different periods, I have had occasion to employ the instrument[.] [A]s the first[,] I had formed a table of the least eight figures of the powers of several numbers[,] and[,] by arranging under each other the successive powers of the same number and by considering the vertical columns[,] I discovered several curious properties amongst others[.] [T]he two following[:]

> An indefinite number of integer values of x may be found which will render these two expressions integers
>
> $$\frac{7^x-1}{10^a} \text{ and } \frac{3^x-1}{10^a}$$

These at first entirely depended on induction[.] I had observed that when $a=1$ the values which x ought to have were found at regular intervals[,] and[,] as a great many of the first trails I made confirmed this[,] I inferred that they would continue to occur at those intervals indefinitely. I then considered the second vertical column or the case of $x = 1$ and made the same trails and the same inference[.] [T]hen I proceeded to the third[,] fourth and fifth[,] making the same observations and with the same result[,] and[,] satisfied with this induction[,] I concluded that the general truth of the theorem I have stated[.]

[T]he evidence which I then processed that this theorem is a truth depend on the number of particular cases in which I had tried it[,] and I had made several different trials of each value of a. It frequently happens that inductive evidence may be greatly strengthened by other reasonings, and that which I am now about to urge in support of the theorem we are considering will also have in the advantage of furnishing another illustration of a remark I had previously made that in some instances it is possible to infer the necessary truth of a series of operations from a single trial.

In examining the vertical columns it appeared that the figures contained in the first

1	26743	84007
7	87201	88049
49	10407	16343
343	72849	14401
2401	09943	00807
16807	69601	05649
17649	87207	
23543	10449	
63801	73143	
53608	12001	
75249		

recurred in periods of four[,] and in fact it is impossible for them not to recur in periods of ten or less than ten, because there are only ten*[20] numerals and after they have all occurred once the next figure must be the same with one of the former[,] and[,] if any one of those which have proceeded occur all[,] which follow it must necessarily be the same as those which successively followed it before because in both cases they results from multiplying the same number continually by 7.

If we now consider the second vertical column[:] since there can only be ten different figures and since each of these can be combined with any one of the four different ones in the first column[,] there cannot be more than forty different pairs[,] and[,] if we consider any one of them as 49[,] this same pair must reappear within forty rows[.] [I]t does in fact reappear in four[,] and[,] when it has once recurred[,] all the following pairs must necessarily be the same as those which followed it before because they are produced by the same operation[.]

This reasoning may easily be applied to all the succeeding vertical columns[,] and hence we deduce this conclusion[:] that if any given combination of figures occur in the first period[,] they will be repeated indefinitely at stated intervals[,] and[,] if they are not found in the first period[,] it is impossible for them to be met with in any subsequent part[.]

The arguments which has just been stated gives to one part of the proposition demonstrative evidence[.] [F]or from it it follows that if for any individual value of a (that for example of $a = 3$)[,] we can find one value of x which satisfies the equation $\frac{7^x - 1}{10^3}$[,] or[,] in other words if we can find some one power of 7 which ends with the figures 001 then we can find any number[.]

[20]* of course zero is included in this number

The other part of the proposition[,] that one value can be found for any value of a[,] still rests entirely on induction.

[Alternate Argument]

Language without the aid of signs is but ill adopted for demonstrating mathematical truths[.] [N]early a page of words have just been employed to prove the periodic recurrence of certain terminal figures[.] [T]his may be accomplished by the help of signs in a tenth part of the space and less exertion of thought[.]

For if there is one value of x which satisfies the equation

$$\frac{7^x - 1}{10^a} = \text{ whole number}$$

let the value of x be i then we have

$$\frac{7^i - 1}{10^a} = w \text{ or } 7^i = 10^a w + 1$$

raise both sides to the power u we have

$$7^{iu} = (10^a w + 1)^u$$

and making $iu = x$ we have

$$\frac{7^x - 1}{10^a} = \frac{(10^a w + 1)^u - 1}{10^a}$$

[T]his last expression is evidently divisible by 10^a whatever be the value of u.

Many other theorems which first presented themselves in this examination have since been strengthened by reasoning of the nature just quoted[,] and some of them have received demonstrative evidence or have been reduced to depend on the truth of some single trail[.] [A]mongst these latter is the proposition that

$$\frac{3^{10^x} - 1}{10^{x+1}} \text{ is always a whole number when } x > 1$$

let us suppose it possible that some value as $x = i$ will make it a whole number then

$$\frac{3^{10^i} - 1}{10^{i+1}} = w \text{ or } 3^{10^i} = 10^{i+1} w + 1$$

raise both sides to the power 10^{x-i} then we find

$$\left(3^{10^i}\right)^{10^{x-i}} = 3^{10^x} = \{10^{i+1} w + 1\}^{10^{x-i}}$$

and

$$\frac{3^{10^x} - 1}{10^{x+1}} = \frac{\{10^{i+1} w + 1\}^{10^{x-i}} - 1}{10^{x+1}} = \frac{10^{(i+1)w^{x-i}} w^{10^{x-i}} + \&c + \frac{10^{x-i}}{1} 10^{i+1} w + 1 - 1}{10^{x+1}}$$

[T]he two units destroy each other[,] and the last term which contains the lowest power of 10 provided[,] x is greater than i[,] has that number raised to the power $x - i + i + 1 = x + 1$[.] [E]very term is therefore divisible by 10^{x+1}. The proposition is therefore demonstrated of all values of x greater than i[.]

[W]e must now by trail find some one individual value which will fulfil the equation[.] [T]he least is $i = 2$ for $3^{100} - 1$ has its last figures three zeros.

[Babbage's Example No. 2: Power of Number]

These properties of number which have been selected for examples are of such a class that they are susceptible of being confirmed or even demonstrated by other processes[.] There were however other which I noticed in the vertical of the same enquiry which seem of a very different nature[,] and[,] as they appear to point out a train of very remarkable properties[,] I shall give a brief sketch of them[,] promising that they rest solely in the evidence of induction.

In order to explain them let us arrange the powers of 4 under each other[[21]][,] and[,] as we have only occasion for the few last figures[,] the other may be omitted[.]

Table 1: Table of 4th Powers

1	4
2	16
3	64
4	256
5	1024
6	4096
7	16384
8	65536
9	262144
10	1048576
11	4194304
12	16777216
13	67108864
14	268435456
15	1073741824
16	4294967296
17	17179869184
18	68719476736
19	274877906944
20	1099511627776
21	4398046511104
22	17592186044416
22	70368744177664
23	281474976710656

On inspection it appears that the first vertical column consists of periods of two figures 4[,] 6 continually repeated[.]

The next observation I make is that the first figure which occurs in the second vertical column is 1 and that it is on a line with 6 in the first column[.] [O]n comparing this with the figure immediately before the 6 which occurs in the second period[,] which is 5[,] I find that 4 must be added to it to produce that figure. [T]his number 4 which must be so added I denote thus $D_1^1 4 = 4$[.] [I]f 5 had been the number whose powers has been considered[,] it would have been written $D_1^1 5 = 4$[.]

Making the same remark with regard to the second figure 6 in the same column[,] I find we must add six to it in order to produce 2 (the tens being negated) which is opposite 4 in the second period this denote by $D_2^1 4 = 6$.

[21] [See table of 4th powers.]

The third observation I make is that if we add 4 to the first figure of 16 it becomes 56[,] the figures which are found at the second period[.] [I]f we again add 4 to the first figure it becomes 96[.] [T]hese figures are found at the third period[,] and again adding to the first figure it becomes 36[,] the figures found at the fourth period[.]

[F]rom this induction I conclude that at the end of any number of periods v the penultimate figure will be $1 + 4v$ rejecting the tens and hundreds if there are any[.] [N]ow if this conjecture is true I observe that when $v = 5$ or 10 or 15 &c that the penultimate figure is 1[.] [T]his 1 will occur every fifth number[,] and[,] as the period of the first column is of two terms[,] it appears that the two last figures must recur at the end of every $5 \cdot 2 = 10$ terms from each other[.] [O]n examining the table it will be perceived that this is actually the case.

The same remark may be made of the figure 64 except that 6 must be added to the first instead of 4[.]

Having established and verified the period of the second column if we examine all the rest of it[.] [I]t will be found that the above observations are entirely correct[,] and[,] by their means[,] it would be easy to assign the two last figures of any power of 4[.]

If we now consider the third column we may remark that 2[,] which is the first figure of 256[,] is less than 4 which precedes 56 in the next period that to make it equal we must add 2[.] [T]his I denote by $D_1^2 4 = 2$[.] [B]y similar considerations we shall find that $D_2^2 4 = 8$[,] $D_2^3 4 = 2$[,] $D_2^4 4 = 8$[,] $D_5^2 4 = 2$ &c and it may also be remarked that if multiples of two are continually add to the first figure of 256 we shall have 456, 656, 856, &c which are the very figure that occur in the succeeding periods[.] [H]ence $2 + 2v$ being the form of the first figure if $v = 5$ it becomes 2 consequently after five periods the same three figures 256 will recur[,] and[,] as this is combined with the period of two figures which recurs at the end of ten rows[,] we conclude that the figures 256 will reappear at the end of fifty rows[,] or[,] since they first occur at the fifth power[,] we should expect to meet with them again at the fifhy fifth on referring to that power we find that they do occur there[.]

It is unnecessary to repeat this reasoning which applies to all the successive columns. I have found the following values for $D_1^1 4$, $D_2^1 4$ &c

$$\begin{array}{llll}
D_1^1 4 = 4 & D_2^1 4 = 6 & & \\
D_1^2 4 = 2 & D_2^2 4 = 8 & D_3^2 4 = 2 & D_4^2 4 = 2 \text{ \&c} \\
D_1^3 4 = 4 & D_2^3 4 = 6 & D_3^3 4 = 4 & D_4^3 4 = 6 \\
D_1^4 4 = 2 & D_2^4 4 = 8 & D_3^4 4 = 2 & D_4^4 4 = 8 \\
D_1^5 4 = 6 & D_2^5 4 = 4 & D_3^5 4 = 6 & D_4^5 4 = 4
\end{array}$$

These quantities appear also to be connected by the following laws[22]

$$\begin{array}{lllll}
D_1^1 4 = 4 \not\approx 4^1 & D_1^2 4 = 2 \not\approx 2 \cdot 4^0 & D_1^3 4 = 4 \not\approx 4^1 & D_1^4 4 = 2 \not\approx 2 \cdot 4^0 & D_1^5 4 = 6 \not\approx 4^2 \\
D_2^1 4 = 6 \not\approx 4^2 & D_2^2 4 = 8 \not\approx 2 \cdot 4^1 & D_2^3 4 = 6 \not\approx 4^2 & D_2^4 4 = 8 \not\approx 2 \cdot 4^1 & D_2^5 4 = 4 \not\approx 4^3 \\
 & D_3^2 4 = 2 \not\approx 2 \cdot 4^2 & D_3^3 4 = 4 \not\approx 4^3 & D_3^4 4 = 2 \not\approx 2 \cdot 4^2 & D_3^5 4 = 6 \not\approx 4^4 \\
 & D_3^2 4 = 8 \not\approx 2 \cdot 4^3 & D_4^3 4 = 6 \not\approx 4^4 & D_3^4 4 = 8 \not\approx 2 \cdot 4^3 & D_4^5 4 = 4 \not\approx 4^5
\end{array}$$

In examining the powers of other numbers[,] I made similar inductions which I shall put down in order to assist any one who may direct his enquiries to this subject[.] [T]he number three

I raised to very high powers[.] Some of them above the millionth[.] [I]t has the following properties

[22] The $\not\approx$ seems to mean "equals mod 10". For example, $4^2 \not\approx 6$ as $4^2 = 16 = 6 \pmod{10}$.

$D^1_1 3 = 6 \not\approx 2{\cdot}3^1$	$D^2_1 3 = 2 \not\approx 2{\cdot}3^0$	$D^3_1 3 = 4 \not\approx 2^2{\cdot}3^0$	$D^4_1 3 = 3 \not\approx 3^1$	$D^5_1 3 = 7 \not\approx 3^3$
$D^1_2 3 = 8 \not\approx 2{\cdot}3^2$	$D^2_2 3 = 6 \not\approx 2{\cdot}3^1$	$D^3_2 3 = 2 \not\approx 2^2{\cdot}3^1$	$D^4_2 3 = 9 \not\approx 3^2$	$D^5_2 3 = 1 \not\approx 3^4$
$D^1_3 3 = 4 \not\approx 2{\cdot}3^3$	$D^2_3 3 = 8 \not\approx 2{\cdot}3^2$	$D^3_3 3 = 6 \not\approx 2^2{\cdot}3^2$	$D^4_3 3 = 7 \not\approx 3^3$	$D^5_3 3 = 3 \not\approx 3^5$
$D^1_4 3 = 2 \not\approx 2{\cdot}3^4$	$D^2_4 3 = 4 \not\approx 2{\cdot}3^3$	$D^3_4 3 = 8 \not\approx 2^2{\cdot}3^3$	$D^4_4 3 = 1 \not\approx 3^4$	$D^5_4 3 = 9 \not\approx 3^6$
	&c &c	&c &c	&c &c	&c &c

Some of the properties of 6 are

$$\begin{array}{lll} D^1_1 6 = 8 & D^2_1 6 = 4 & D^3_1 6 = 2 \\ & D^2_2 6 = 4 & D^3_2 6 = 2 \\ & D^2_3 6 = 4 & D^3_3 6 = 2 \\ & D^2_4 6 = 4 & D^3_4 6 = 2 \\ & & \&c \end{array}$$

[F]or the number 7[,] I have found

$$\begin{array}{lll} D^1_1 7 = 0 & D^2_1 7 = 2 \not\approx 2^1 & D^3_i 7 = D^4_i 7 = D^5_i 7 = \&c \\ D^1_2 7 = 0 & D^2_2 7 = 4 \not\approx 2^2 & = D^2_i 7 \not\approx 2^i \\ D^1_3 7 = 0 & D^2_3 7 = 8 \not\approx 2^3 & \\ D^1_4 7 = 0 & D^2_4 7 = 6 \not\approx 2^4 & \\ & \&c \quad \&c & \end{array}$$

The number 5 posess the most singular properties of any I have examined probably from its being half the radix of the decimal system. The last two figures are always 25[.] [T]he three last figures reappear at the end of 2 rows[.] [T]he 4 last reappear at the end of 2^2 rows and generally the n last figures reappear at the end of 2^{n-2} rows: also $D^1_1 5 = 0$ and universally when n is greater than 1 $D^n_i 5 = 5$[.]

[Babbage's Example No. 3: Cube Roots]

The other instance in which I had recourse to induction was to find some rule for extracting the cube root of a perfect cube of less than ten placed of figures. The celebrity of an American child in performing such operations without the assistance of paper had excited some attention to the subject. It is needless to repeat the inductive process[.] [T]he rule arrived at was as follows.

If the given cube ends in an even number it may be continually divided by 8 untill it ends with an odd one[.]

If it ends with 5 it may be divided by 125[.]

These division being performed if necessary a perfect cube will remain[.]

The root of the first period is known by inspection[.] [T]his gives the first figure of the root[.]

Cube the last figure and the number with which it ends is the last figure which must be either 1, 7, 3, or 9[.]

In order to find the middle figure v determine it from one of the following equiations[.]

[I]f the last figure of root is	1[,] the equation is	$3 + 3v =$	last fig.	but one of	giv. cube
——————————-	3 —————	$2 + 7v =$	D^0	D^0	D^0
——————————-	7 —————	$4 + 7v =$	D^0	D^0	D^0
——————————-	9 —————	$2 + 3v =$	D^0	D^0	D^0

[Constructing Series where Enumerative Induction Fails]

I have already mentioned a case in which Euler[,] from an induction founded on ten cases[,] deduced a law which was incorrect[.] [I]nstances might be produced in which much larger number of cases would not be more successful[.] [S]hould any one make trial of the formula $x^2 + x + 41$ in order to find whether it does not always express prime numbers[,] he would find that for the first forty numbers the results would all be primes[.] [Y]et it is evident all numbers contained in this formula are not primes because whenever x is a multiple of 41[,] the resulting number must be be divisible by 41.

To place in a strong point of view the danger of trusting to induction evidence[,] I shall show that even in a very simple case[,] two expressions may coincide with each other for any given number of terms and yet fail at some other[.]

Let ${}^nS_i^1$ denote the sum of the ith powers of the nth roots of unity divided by n[.] [T]hen $i +^n S_i^1$ will be the general term of the series

$$1, 2, 3, ...n - 1, n, n + 1, n + 2, n + 3, ...\&c$$

and this coincides with the supposition that the general term is i for the first $n - 1$ cases but it fails in the nth.[23]

[Power Series by Differentiation: Analogy Between Powers and Differentials]

The inventive faculty[,] consisting pobably of the combined operation of several others, it is not surprising that it should be difficult to produce examples which illustrate one only of whose principles by which it appears to act[.]

It has been observed by M. Lacroix[,] of the memoir which contain the reasoning on the subject which I have chosen as the next instance of induction[:]

> ”That can be regarded as one of the most beautiful application that we have made of the method of induction.”†[24]

[Y]et the same theorem might without propriety be adduced us examples either of generalisation or of analogy. As the subject is one of curiosity both from the celebrity of those who have employed it, and to the brilliant results[25] which Lagrange deduced from it[,] the reader will excuse the detail into which I shall enter in order to place in a clear point of view the train of thought which appears to have conducted to it.

In the *Reports of the Scholars*[26] *[27] 1694 p. 437[,] John Bernoulli[28] had in-

[23] Babbage’s explanation here is a bit brief. An expanded explanation is found in Dubbey 1978 *The Mathematical Works of Charles Babbage* pp. 111–2.

[24] † [The qoute is from Lacroix’s *Analytical Theory of Probabilities* (*Théorie analytique des probabilités*).] [Quote translated by Franck Chauvel from: ”Quélle peut être regardé comme une des plus belles applications que l’on ait faites de la méthode des inductions.” Lacroix is in the quote referring to ”Sur une nouvelle espèce de calcul. pour M. Lagrange Mém. Acad. de Berlin 1772 (1774)”]

[25] e.g. Series Expansion, Lagrange Reminder

[26] Translated by Wikipedia from *acta Eruditorum*

[27] * see also John Bernoulli *Opera omnia* [Vol.] I [p.] 125

[28] This is Johann Bernoulli, 1667–1748. He is also known as Jean or John; Babbage uses the latter.

vestigated the series[29][30,31,32:]

$$\int u \cdot dx = \frac{x}{1}u - \frac{x^2}{1 \cdot 2}\frac{du}{dx} + \frac{x^3}{1 \cdot 2 \cdot 3}\frac{d^2u}{dx^2} + \&c$$

in a manner which had excited the admiration of his acute friend and correspondent Leibnitz[,] who[,] in a postcript to a letter which is dated Feb, 1695[,] attempts to make use of similar reasoning for a different series[.]

> P.S. Although I had clearly decided to take a short break from analytical reflections for the sake of my health, I was however unable to to refrain from giving greater consideration to that most attractive method whereby you have investigated the general series. On doing this I saw that such a series is to be had by a similar procedure if one does away with one end. [33]

An accidental error in the calculation gave an incorrect result which was corrected in the reply of John Bernoulli who remarked

> The deductions you have made in accordance with my method of investigating the general series are excellent; it is enough for me if my discoveries, however trivial they be, provide great men with the opportunity of proceding to greater things. Meanwhile I find an error in your calculation, which without doubt you committed in your haste[.34]

The series of Leibnitz when corrected is[35:]

$$\int z^e \cdot d^m \cdot u = z^e \cdot d^{m-1} \cdot u - e \cdot z^{e-1} \cdot d^{m-2} \cdot u \cdot dz + e \cdot (e-1) \cdot z^{e-2} \cdot d^{m-3} \cdot u \cdot (dz)^2$$

$$-e \cdot (e-1) \cdot (e-2) \cdot z^{e-3} \cdot d^{m-4} \cdot u \cdot (dz)^3 + \&c$$

This series assumes a resemblance to that of the eth power of a binomial[,] and[,] with probability of this circumstance[,] we are indebted for the discovery which Leibnitz now made of the beautiful analogy that subsists between powers and differentials[.]

In his next letter to John Bernoulli he observes.

29 The Bernoulli Series. Can be found by repeatedly applying integration by parts, or, as Bernoulli actually did it, as described in Ferraro 2008, p. 45.

30 Edwards, *The Historical Development of the Calculus*, p. 290: "In the *Acta Eruditorum* of 1694 John Bernoulli published a series that was sufficiently similar to Taylor's for Bernoulli to accuse Taylor of plagiarism when the *Methodus incrementorum* appeared twenty yeats later."

31 Jahnke, *A History of Analysis*, p. 111: "An interesting predecessor of Taylor's theorem was published in 1694 by Johann Bernoulli in the Acta eruditorum."

32 Taylor proved his theorem by listing a few cases and concluding the similarity to binomial expansions. Bernoulli and Leibniz did the same. It can also be arrived at by incremental application of integration by parts and noting the similarity to binomial expansions.

33 Translated by Quintus' Latin Translation Service from: "P.S. Tametsi plane constituissem temperare mihi nonnihil valetudinis causa ab analyticis meditationibus, non potui tamen impetrare a me, quin pulcherrimam illam rationem, qua seriem generalem indagasti, considerarem attentius. Quo facto vidi, altero termino destructo, simili methodo talem seriem haberi[.]"

34 Translated by Quintus' Latin Translation Service from "Egregia sunt quae ex ratione mea seriem generalem indagandi deduxisti; mihi sufficit; si inventa mea, ut ut tenuia, magnis viris occasionem dederint ad majora: Interim in calculo tuo lapsum reperio, quem haud dubie praecipitanter commiseris"

35 p. 347. This formula was quite messy in Babbage's manuscript. The original was looked up instead.

> ”Well corrected calculation ..... There are yet many things latent in these progressions of summation and differentiation, which will gradually appear. There is thus notably agreement between the numerical powers of binomial and differential expansions; and I believe that I do not know what is hidden there.” [36]

Leibnitz then compares the four first powers of the binomial $x + y$ with the four first differentials of xy[37:]

$$\begin{array}{cc} x+y & ydx+xdy \\ x^2+2xy+y & yd^2x+2dydx+xd^2y \\ x^3+3x^2y+3xy^2+y^3 & yd^3x+3dyd^2x+3d^2ydx+xd^3y \\ x^4+4x^3y+6x^2y^2+4xy^3+y^4 & yd^4x+4dyd^3x+6d^2yd^3x+4d^3ydx+xd^4y \end{array}$$

and observes that the analogy is complete[.] [F]or that the latter expression may always be deduced from the former by assigning $x^m y^n$ into $d^m x d^n y$ as for example $x^2 = x^2 y^0$ then becomes $d^2 x d^0 y = y d^2 x$ since $d^0 y = y$[.] [H]e extends the same remarks to the analogy between $(x + y + z)^m$ and $d^m xyz$ and also to polynomials in general.

This instance of induction[,] resting on but few cases[,] is never the less of great weight. The reason which compels our assent to it appears to be that we cannot discover any new cause which may come to play in the higher powers which shall disturb that regularity that is apparent in the lower ones[.] [T]here is in fact nothing peculiar in the nature of the exponent on which we can surmise that this property depends[.] [H]ad the exponent been restricted to prime numbers[,] for example[,] so small a number of coincidences would hardly have satisfied the most careless enquirer.*[38]

It is rather a curious fact that Leibnitz[,] who observers in this letter that $d^0 y = y$[,] would not go on to the parallel case of integrals and negative powers[,] and yet that he should have some view [39] respecting fractional indices of differentiation[.] [S]uch however appears to be the case from the following paragraph[:]

> ”Furthermore, we must see whether in summations (integrations) one may conceive of something that corresponds to irrational, or even contrived, roots.” [40]

In the reply of John Bernoulli to this letter[,] after expressing his admiration at the elegant analogy between powers and differentials which Leibnitz has pointed out[,] he pushes it a little further and proposes treating the sign as a generality[:]

> ”Consider the d, d^2, d^3, d^4 etc. as algebraic quantities and letters not only caracteristicis.[”41]*[42][.]

[36] Translated by Harold T. Davis from: ”Recte correxisti calculum ..... Multa adhuc in istis summarum et differentiarum progressionibus latent, quar paulatim prodibunt. Ita notabilis est consensus inter numeros potestatum a binomio, et differentiarum a rectangulo et puto nescio quid arani subesse.”

[37] p. 353. These are implicit differentials.

[38] Babbage wrote a star but no corresponding footnote.

[39] [I]n the peculiar mode which Bernoulli alluded to[,] all the indices were to be transferred from the quantities to the symbols of operation [?] [.]

[40] Translated by Quintus’ Latin Translation Service from: ”Imo videndum, an non in summationibus (integrationibus) concipere aliquid liceat respondens radicibus irrationalibus, imo affectis.”

[41] Translated by Lenore Feigenbaum from: ”consideratis interim d, d^2, d^3, d^4 etc. tanquam quantitatibus algebraicis et non literis tantummodo caracteristicis.”

[42] *How short a step separated this from the process of Arbogast [Louis François Antoine Arbogast (1759–1803)] of separating operations from quantities on which they act!

[F]rom such considerations he deduced the integral of an equation and converts another into a series; from which he thus breaks off.

> ["]I see that at this point, as I write, something has unexpectedly happened to the universal method of summing a differential quantity of any degree, either by means of, or without employing, a series; I also see that other infinite matters still lie hidden here."[[43]]

Thus[,] which in the act of writing to his friend[,] his invention was at work[.] [I]n this instance[,] the result was rather different from that he had anticipated. [A]s we find in his next letter[:]

> Let it be what we are to seek out $\int ndz$; let it be differentiated ndz, it will be had $nddz + dndz$; therefore, by my method, a third proportional[[44]] of the same must be taken $d^0nddz + dndz$ to d^0ndz, which in this way will be $\frac{d^0nddz}{d^0nddz+dndz}$ = (by dividing the numerator and denominator by dz) $\frac{d^0ndz}{d^0ndz+dnd^{0n}}$: when a continuous division is made, beginning from the earlier component of the denominator, [[45]]
>
> $$\int ndz = d^0nd0z - dnd^{-1}z + d^2nd^{-2}z - d^3nd^{-3}z\&c =$$
>
> $$nz - dn\int z + d^2n\int^2 z - d^3n\int^3 z\&c.$$
>
> arises, but when division begins from the latter component, it will be
>
> $$\int ndz = d^{-1}ndz - d^{-2}nddz + d^{-3}nd^3z - d^{-4}nd^4z\&c =$$
>
> $$dz\int n - d^2z\int^2 n + d^3z\int^3 n - d^4z\int^4 n, \&c$$
>
> Now because (if dz is supposed constant) $\int z, \int^2 z, \int^3 z, \int^4 z, \&c$ are equal to
>
> $$\frac{zz}{1{\cdot}2{\cdot}dz^1}, \frac{z^3}{1{\cdot}2{\cdot}3dz^2}, \frac{z^4}{1{\cdot}2{\cdot}3{\cdot}4dz^3}, \frac{z^5}{1{\cdot}2{\cdot}3{\cdot}4{\cdot}5dz^4}, \&c$$
>
> The earlier series
>
> $$\int ndz = nz - dn\int z + d^2n\int^2 z - d^3n\int^3 z\&c$$
>
> will be converted into this
>
> $$\int ndz = nz - dn\frac{z^2}{1{\cdot}2dz} + d^2n\frac{z^3}{1{\cdot}2{\cdot}3dz^2} - d^3n\frac{z^4}{1{\cdot}2{\cdot}3{\cdot}4dz^3}, \&c.$$

[43] Translated by Quintus' Latin Translation Service from: "Video me hic inter scribendum et quidem ex insperato incidisse in methodum universalem summandi vel per vel citra seriem quantitatem differentialem cujuscunque gradus; video etiam infinita alia adhucdum abscondita hic latere."

[44] What he is here doing is saying that $\frac{d^{-1}(ndz)}{d^0(ndz)} = \frac{d^0(ndz)}{d^1(ndz)}$. Then he rearranges to get $d^{-1}(ndz)$, which is $\int ndz$, on the left side. This is called taking the *third proportional*.

[45] From Ferraro 2008: "Since Mercator's rule led one to find two different expansions of a quantity $\frac{a}{b+c}$, Bernoulli obtained two different expansions $\int ndz$ ..."

> which is completely the same as the one I published in the Proceedings; I am greatly amazed by this; for this is the outcome, when I began to write these things, I did not indeed expect this result, thinking that I would arrive at a far different series by this method. This elegant agreement wonderfully confirms the probity of the methods, especially of this last, where so remarkably and contrary to all practice it is advanced with the letters d. Thus even now I am of the opinion that other infinite and unheard-of things can be unearthed from this, as long as someone is willing to pursue these matters with more detailed scrutiny. [[46]]

The next letter of Leibnitz which is dated October 1695[,] about eight months after the communication which referred to these singular analogies[,] contains the first explicit statements that they apply with slight change to integrals as well as to differentials.

> Since, as you know, the differences are analogous to the powers, hence from the series for the powers I drew the series for the differences, in this way:
>
> $$(x+y)^m = x^m y^0 + \frac{m}{1} x^{m-1} y^1 + \frac{m(m-1)}{1{\cdot}2} x^{m-2} y^2, \&c$$
>
> Therefore
>
> $$d^m xy = d^m x d^0 y + \frac{m}{1} d^{m-1} x d^1 y + \frac{m(m-1)}{1{\cdot}2} d^{m-2} x d^2 y, \&c$$
>
> where, turning d into $\int$, so that $d^m = \int^m$ with $n = -m$, then
>
> $$\int^n dzy = \int^{n-1} zd^0 y - \frac{n}{1} \int^n zd^1 y + \frac{n(n+1)}{1{\cdot}2} \int^{n+1} zd^2 y$$
> $$-\frac{n(n+1)(n+2)}{1{\cdot}2{\cdot}3} \int^{n+2} zd^3 y, \&c$$
>
> where, having posited dz a constant, the sums individually can be entered upon, and indeed finitely, if n it is an integer. One may devise similar things instead of a trinomial or other polynomials, and contrive other analogies of all kinds. [[47]]

From this period both Leibnitz and J[ohn] Bernoulli appear to have laid aside[,] for many years[,] any further investigations concerning this analogy[.]

[Miscellanea Berolinensia Example]

In the first volume of the *Miscellanea Berolinensia*[,] a work which Leibnitz was very instrumental in establishing[,] we meet with a paper of his on the subject which has this title[:] "A memorable symbolism of Algebraic and Infinitesimal calculus in the comparison of powers and differences."[[48].]

[46] Translated by Quintus' Latin Translation Service, H. J. M. Bos and Lenore Feigenbaum from Quote 1 (see the end of the document).

[47] Translated by Quintus' Latin Translation Service and Lenore Feigenbaum from Quote 2 (see the end of the document).

[48] Translated by Quintus' Latin Translation Service from "Symbolismus memorabilis calculi Algebraici et Infinitesimalis in comparatione potentiarum et differentiarum"

In this paper[,] in order to make the resemblance more striking[,] he denotes the eth power of x thus $p^e x$[.] [B]ut the most important alteration consisted in the nature of the proof by which he supported the propositions[.] [I]n the former cases it rested merely on inductive evidence[,] but in this paper he shows from the nature of the *[49] operation of differentiation that the analogy must necessarily be preserved and therefore that the coefficients *must* be the same as those of the binomial[.] [T]he same analogy is shown to exist for multinomials[.]

[I]t is rather singular that no mention is made of integrals and that Leibnitz should express as astonishment at it holding true for the case of d^0[.] [F]or he observes,

> "And this analogy even goes so far that, in this way of notation (which may surprise you), also $p^0(x+y+z)$ actually corresponds to $d^0(zyx)$[.]" [50]

[Lagrange Example]

After Leibnitz and J[ohn] Bernoulli[,] no geometer added to what was already known concerning these analogies untill the year 1772 when Lagrange made them the subject of a paper in the *New Memoirs of the Royal Academy of Sciences and Belles-Lettres of Berlin*[51.] The mode of explaining the differential calculus[,] which he there adopts[,] embraces the principles which afterward found the basis of his *Theory of Analytical Functions*[52].

His first step is to prove that if u is a function of $x, y, z, \&c$ and if these quantities become $x+\xi, y+\psi, z+\zeta$ &c then we shall have

$$\Delta u = e^{\frac{du}{dx}\xi + \frac{du}{dy}\psi + \frac{du}{dz}\zeta + \&c} - 1$$

provided we always change in the development du^λ into $d^\lambda u$[.][53]

[S]o far his reasoning is legitimate but the next step is to raise both sides of this equation to the power λ and to conclude that the equation

$$\Delta^\lambda u = \left(e^{\frac{du}{dx}\xi + \frac{du}{dy}\psi + \frac{du}{dz}\zeta + \&c} - 1\right)^\lambda$$

is also true subject to the above condition[.] [F]rom this by supposing λ negative he deduces the value of $\Sigma^\lambda u$.

[49]* ["]This analogy between differentiation and potentiation is preserved in perpetuity, when the potentiation (or execution of the Power) and differentiation has been continued. Of course in the new potentiation of a Binomial the preceding is multiplied in its entirety both by y and by x etc., and it is increased by a unit in the former case p of the same y, and in the latter case p of the same x.["] [Translated by Quintus' Latin Translation Service from "Quae analogia inter differentiationem & potentiationem servatur perpetuò, continuata potentiatione [seu Potentiae executione] & differentiatione. Nempe ut in nova potentiatione Binomii totum praecedens multiplicatur tam per y quam per x, &c priore casu p ipsus y, posteriore p ipsus x, augetur unitate[.]"

[50]Translated by H. J. M. Bos from "Eaque analogia eousque porrigitur, ut tali scribendi more (quod mireris) etiam $p^0(x+y+z)$ & $d^0(zyx)$ sibi respondeant, & veritati[.]"

[51]Babbage wrote: *Mémoires de Acad. de Berlin.* It was inferred from the rest of the section to be *New Memoirs of the Royal Academy of Sciences and Belles-Lettres of Berlin* which is translated by the Euler Archive from *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*. The article Babbage is refering to is "On a New Kind of Calculation Related to the Differentiation and to the Integration of Varying Quantities" by Lagrange from 1772 translated by Franck Chauvel from "Sur une nouvelle espece de calcul rélatif à la différentiation et à l'intégration des quentités variables".

[52]Babbage wrote "Théorie des Functions". This is probably *Theory of Analytical Functions* from 1797 by Lagrange, whose title was translated by Craig G. Fraser from *Théorie des fonctions analytiques*.

[53]Note that the series expansion of $e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \ldots$.

The defective nature of the proof was acknowledged by this great geometer[,] who appeared to put confidence in it partly from its analogy to other theorems[,] but chiefly from its affordings in many particular cases a variety of theorems whose truth was known by other means[.] [O]f this coincidence[,] he produces many examples in the course of his paper.

> "Allthough the operation with which we transition from the difference Δu, to the difference $\Delta^{\lambda} u$ and to the sum $\Sigma^{\Lambda} u$ is not really grounded on rigorous principles, it is still correct as one can check *a posteriori* but it might be very difficult to give an exact analytical explanation; this is due to the general analogy that exist between the positive exponent and the differentiations as well as between the negative exponents and the integration; an analogy of which we will see many examples in the remainder of this memoire."[[54]]

So far then as regarding the principle steps by the reasoning contained in the paper referred to it may be regarded as a bold but fortunate conjecture and the evidence in its favour was its exact coincidence with well known truths in a variety of particular instances[.]

[Sum of Squares and Sum of Higher Powers]

Few works afford so many examples of pure and unmixed induction as *The Arithmetic of Infinitesimals*[55] [of John Wallis (1616–1703)][,] and[,] although more rigid methods of demonstration have been substituted by modern writers[,] this most original production will never cease to be examined with attention by those who interest themselves in the history of analytical science or in examining those trains of thought which have contributed to its perfection.

The ration which the sum of any number of terms of the of series of square numbers (commencing with zero) has to as many times the greatest term is thus discovered[.]

Wallis prescribes[:[56]]

> The investigation may be done by the method of induction. And we have:

$$\frac{0+1}{1+1} = \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$

[54]Translated by Franck Chauvel from: "Quoique l'operation par laquelle nous avons passé de la difference Δu, à la difference $\Delta^{\lambda} u$, & à la somme $\Sigma^{\Lambda} u$, ne soit pas fondée sur des principes claires & rigoureux, elle n'en est cependant pas moins exacte, comme un s'en assurer *a posteriori*; mais il seroit peut-être très difficile d'en donner une démonstration directe & analytique; cela tient en général à l'analogie qu'il y a entre les puissances positives & les différentiations, aussi bien qu'entre les puissances negative & les intégrations; analogie dont nous verrons encore d'autres exemples dans la suite de ce Mémoire." quote from Lagrange's "On a New Kind of Calculation Related to the Differentiation and to the Integration of Varying Quantities" whose title is translated by Franck Chauvel from "Sur une nouvelle espece de calcul rélatif à la différentiation & à l'intégration des quantités variables".

[55]Translated title by Jacqueline A. Stedall from "Arithmetica Infinitorum"

[56]John Wallis, Arithmetica Infinitorum, 1656, page 15. Translated by Jacqueline A. Stedall from "Fiat investigatio per modum inductionis"

$$\frac{0+1+4+9}{9+9+9+9} = \frac{14}{36} = \frac{7}{18} = \frac{1}{3} + \frac{1}{18}$$

$$\frac{0+1+4+9+16}{16+16+16+16+16} = \frac{30}{80} = \frac{3}{8} = \frac{9}{24} = \frac{1}{3} + \frac{1}{24}$$

$$\frac{0+1+4+9+16+25}{25+25+25+25+25+25} = \frac{55}{150} = \frac{11}{30} = \frac{1}{3} + \frac{1}{30}$$

The ratio approaches more nearly at every step to that of 1 to 3 from which it only differ by a fraction whose numerator is unity and denominator some multiple of six[.] If therefore the number of terms be augmented indefinitely[,] this ratio becomes correct within any assignable limits[.]

This proposition immediately determines the ratio of a cone to its circumscribing xylinder[,] or that of the area of a parabola to the rectangle in which it is inscribed[.] These propositions it is true were well known long previous to the enquiries of Wallis[,] but they formed a natural introduction to others of a singular kind in which he had not been anticipated[.]

The same inductive process he applied to series of powers whose exponents are the numbers 3, 4, 5 &c and from the resulting propositions the quadrature of most of the higher species of parabolas easily followed.

[Towards a Formula Containing All Primes]

Our ignorance of any form comprehending all prime numbers adds very justly to our doubts respecting any induction in which their properties may be concerned. It is possible that the formula comprehending them may be of some such form as the following

$$f(x) + \varphi(x) + \varphi_1(x)$$

where the form of f is some simple one and in which the function φ is of such a nature that it only affects the result in very large numbers, and φ_1 may only affect it is others still larger. If such were the case it is easy to conceive that the property supposed to be discovered by induction might in fact depend on the peculiar form of f and it would therefore be verified for all small numbers but must necessarily fail when the term $\varphi(x)$ comes into action[,] or if the properly depended on the mutual relation of f and φ[,] it might be confirmed by a still larger number of instances although ultimately false.

Some time after these remarks were written I endeavoured in some measure to verify them[,] but[,] as we are not acquainted with any process containing only primes[,] I chose one containing very considerable number in order to make any observations on it[.] I propose then to show from probable circumstances that the formula which contains primes is of the form[:]

$$f(x) + \varphi(x) + \varphi_1(x) + \&c$$

f, φ, φ_1 having the property I have stated above[.] I begun by assuming $f(x) = x^2 + x + 41$ which is sufficiently simple and which has the first 40 numbers all primes[.] [A]ll the other functions must therefore vanish when x is is less than 41.

In the next forty primes of x or from $x = 41$ to $x = 80$[,] there occur six numbers which are not primes[.] [T]hey correspond to the following values of x viz. $x =$

$41, 44, 49, 56, 65, 76$[.] [T]hese occur at unequal but yet regular intervals and all the values of x included in them are comprised in the expression $x = 40 + i^2$[.] On examining the whole of the third period on from $x = 81$ to $x = 120$ and several other values of x contained in $x = 40 + i^2$[,] I have in all cases found them to be composite numbers. I am therefore led to conclude that function $\varphi(x)$ must be of such a nature as to vanish for all values of x less than 40 and only to come into play when x is of the form $40 + i^2$[.] [B]y examining the numbers which constitute the third period from $x = 81$ to $x = 120$ and comparing them with the expression $f(x) + \varphi(x)$[,] I find there are only nine which are given by it for primes which are not [?][.] [T]hese nine correspond to the following values of x[,] $81, 82, 84, 87, 91, 96, 102, 109, 117$[,] and I observe that their second difference are constant and that they can all be represented by $80 + \frac{i^2-i+2}{2}$.

Hence if φx be such a function of x that it shall become unity for all values of x contained in $40 + i^2$ and vanish for all others[,] and if $\varphi_1(x)$ be such a function that it shall become unity for any value of x of the form $80 + \frac{i^2-i+2}{2}$ and zero for all others[,] then will the formula

$$x^2 + x + 41 - \{x^2 + x + 41\}\varphi(x) - \{x^2 + x + 41\}\varphi_1(x)$$

contain a great many prime numbers.

In fact the first 105 numbers which it gives are primes. It is possible to determine functions having the prescribed property by measure of the roots of unity for which properties the reader may consult a paper of [M?] in Phil. Trans. on Periodic functions[.]

[Quote 1]

[

> ”Quaerenda esto $\int ndz$; differentietur ndz, habebitur $nddz + dndz$; ergo, modo meo, sumenda est tertia proportionalis ipsius $d^0nddz + dndz$ ad d^0ndz, quae itaque erit $\frac{d^0nddz}{d^0nddz+dndz}$ = (dividendo numeratorem et denominatorem per dz) $\frac{d^0ndz}{d^0ndz+dnd^0n}$: facta divisione continua, inchoando a priori denominatoris membro, provenit
>
> $$\int ndz = d^0nd^0z - dnd^{-1}z + d^2nd^{-2}z - d^3nd^{-3}z\&c =$$
>
> $$nz - dn\int z + d^2n\int^2 z - d^3n\int^3 z\&c.$$
>
> inchoata vero divisione à posteriori membro, erit
>
> $$\int ndz = d^{-1}ndz - d^{-2}nddz + d^{-3}nd^3z - d^{-4}nd^4z\&c =$$
>
> $$dz\int n - d^2z\int^2 n + d^3z\int^3 n - d^4z\int^4 n, \&c$$
>
> quoniam nunc (posita dz constante) $\int z, \int^2 z, \int^3 z, \int^4 z, \&c$ aequantur ipsis
>
> $$\frac{zz}{1{\cdot}2{\cdot}dz^1}, \frac{z^3}{1{\cdot}2{\cdot}3dz^2}, \frac{z^4}{1{\cdot}2{\cdot}3{\cdot}4dz^3}, \frac{z^5}{1{\cdot}2{\cdot}3{\cdot}4{\cdot}5dz^4}, \&c$$

prior series

$$\int ndz = nz - dn\int z + d^2n\int^2 z - d^3n\int^3 z\&c$$

convertetur in hanc

$$\int ndz = nz - dn\frac{z^2}{1{\cdot}2dz} + d^2n\frac{z^3}{1{\cdot}2{\cdot}3dz^2} - d^3n\frac{z^4}{1{\cdot}2{\cdot}3{\cdot}4dz^3}, \&c.$$

quae omnino eadem est, quam in Actis publicavi, quod valdopere miror; hunc enim eventum, cum haec inciperem scribere, non sperabam; putans longe aliam seriem hac methodo proventuram: Elegans iste consensus mirifice methodorum probitatem, praesertim hujus posterioris, ubi tam mirabiliter et contra omnem consuetudinem cum literis d proceditur, confirmat. Sic etiamnum sum opinione, infinita alia et inaudita inde erui posse, dummodo aliquis attentiori scrutatione illa prosequi vellet[”]

]

[Quote 2]

[

”Quoniam, ut scis, potentiis analogae sunt differentiae, hinc ex serie pro potentiis duxi seriem pro differentiis, hoc modo;

$$(x+y)^m = x^m y^0 + \frac{m}{1}x^{m-1}y^1 + \frac{m(m-1)}{1{\cdot}2}x^{m-2}y^2, \&c$$

Ergo fit

$$d^m xy = d^m x d^0 y + \frac{m}{1}d^{m-1}xd^1y + \frac{m(m-1)}{1{\cdot}2}d^{m-2}xd^2y, \&c$$

Ubi vertendo d in $\int$, ut sit $d^m = \int^m$, posito $n = -m$, siet
$\int^n dzy =$
$\int^{n-1} zd^0y - \frac{n}{1}\int^n zd^1y + \frac{n(n-1)}{1\cdot2}\int^{n+1} zd^2y - \frac{n(n-1)(n-2)}{1\cdot2\cdot3}\int^{n+1} zd^3y, \&c$
ubi, posito dz constante, summae singulatim miri possunt et quidem finite, si n integer. Similia pro trinomio vel aliis polynomiis fabricare licet, aliasque omnigenas analogias comminisci.”

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[References]

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- Isaac Newton, *Commercium Epistolicum* (London: 1712).

- Leonhard Euler, "Example of the use of observation in pure mathematics," E256, 1756, *Novi Commentarii academiae scientiarum Petropolitanae*, No. 6, 1761, p. 187.
- Leonhard Euler, "Being carried out by induction with complete certainty", E566, *Acta Academiae Scientarum Imperialis Petropolitinae*, No. 4, 1784, pp. 38–48.
- Leonard Euler, "Various methods for inquiring into the innate characters of series", E551, *Brief analytical works*, vol. 1, Saint Petersburg, 1783, p. 48.
- Leonhard Euler, "Observations Concerning A Certain Theorem Of Fermat And Other Considerations Regarding Prime Numbers", E26, *Commentaries of the St. Petersburg Academy of Sciences*, 1738, pp. 103–107.
- Leonhard Euler, "The Demonstration Of Certain Theorems Regarding Prime Numbers", E54, *Commentaries of the St. Petersburg Academy of Sciences*, 8, 1741, pp. 141–146.
- Pierre Simon Laplace, *Analytic Theory of Probabilities*, 1820.
- Joseph-Louis Lagrange, "On a New Kind of Calculation Related to the Differentiation and to the Integration of Varying Quantities", *New Memoirs of the Royal Academy of Sciences and Belles-Lettres of Berlin*, vol. 3, 1772, p. 185.
- Johann Bernoulli, "Additamentum Effectionis omnium quadraturarum et rectificationum curvarum per seriem quandam generalisssimam", *Reports of the Scholars*, 1694, pp. 437–441.
- Gottfried Leibniz, *Leibnizens gesammelte Werke,* 1847.
- Gottfried Leibniz, "A memorable symbolism of Algebraic and Infinitesimal calculus in the comparison of powers and differences," *Miscellanea Berolinensia ad incrementum scientiarum*, No. 1, 1710, pp. 160–165.
- Joseph Louis Lagrange, *Theory of Analytical Functions* (Paris: 1797)
- John Wallis, *The Arithmetic of Infinitesimals or a New Method of Inquiring into the Quadtrature of Curves, and other more difficult mathematical problems*, 1656, p. 15.
- M[?], "[Article on Periodic functions]" *Philosophical Transactions*.

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[Appendix]

[Babbage's Autobiography and *Of Induction*]

Babbage recalls the writing of *Of Induction* in his autobiography:

> During my residence with my Oxford tutor, whilst I was working by myself on mathematics, I occasionally arrived at conclusions which appeared to me to be new, but which from time to time I afterwards found were

> already well known. At first I was much discouraged by these disappointments, and drew from such occurrences the inference that it was hopeless for me to attempt to invent anything new. After a time I saw the fallacy of my reasoning, and then inferred that when my knowledge became much more extended I might reasonably hope to make some small additions to my favourite science.
>
> This idea considerably influenced my course during my residence at Cambridge by directing my reading to the original papers of the great discoverers in mathematical science. I then endeavoured to trace the course of their minds in passing from the known to the unknown, and to observe whether various artifices could not be connected together by some general law. The writings of Euler were eminently instructive for this purpose. At the period of my leaving Cambridge I began to see more distinctly the object of my future pursuit.
>
> It appeared to me that the highest exercise of human faculties consisted in the endeavour to discover those laws of thought by which man passes from the known to that which was unknown. It might with propriety be called the philosophy of invention. During the early part of my residence in London, I commenced several essays on Induction, Generalization, Analogy, with various illustrations from different sources.
>
> Most of the early essays I refer to were not sufficiently matured for publication, and several have appeared without any direct reference to the great object of my life.

[Dugald Stewart and *Of Induction*]

[Dugald Stewart thanks Babbage on p. 396 of his *collected works*. He gives some of the same quotes as appear in *Of Induction*. He writes:

> I am indebted to Mr. Babbage for the following very curious extracts from Euler, on the subject of Induction in mathematics.—Kinneil, Aug. 1819.

This means that he learned about Babbage's work on this topic at Kinneil at this time, which means Babbage must have had the quotes by then and had done the parts of the study.

Stewart's remarks appear as Note Y, which he references from p. 316 where he discusses the use of induction in mathematics.

Babbage recalls the meeting with Stewart in his autobiography:

> I spent a delightful week at Kinneil with Dugald Stewart. The second volume of his "Philosophy of the Human Mind" had fortunately fallen into my hands at an early period during my residence at Cambridge, and I had derived much instruction from that valuable work.

Philosophy of the Human Mind was published as three volumes, in 1792, 1814 and 1827. The second edition of volume 2 was published in 1816, and does not include Babbage's notes. This must have been added to a later edition.]

[*The Arithmetic of Infinitesimals* and *Of Induction*]

[*Of Induction* is quoted in the preface of the translated edition of Wallis' The Arithmetic of Infinitesimals (p. xxxiii):

Almost two centuries after the Arithmetica infinatorium was written, in 1821, Charles Babbage in an unpublished essay entitled 'Of Induction' wrote:

> Few works afford so many examples of pure and unmixed induction as the *Arithmetica infinatorium* of Wallis and although more rigid methods of demonstration have been substituted by modern writers this most original production will never cease to be examined with attention by those who interest themselves in the history of analytical science or in examining those trains of thought which have contributed to its perfection.

]

[John Wallis on Induction]

[John Wallis was Newton's mathematics teacher. Wallis has been given partial credit for inventing calculus.

According to Florian Cajori article "Origin of the Name "Mathematical Induction"" from 1918, there are two sides in the discussion of induction in mathematics. He says "He [John Wallis] ... freely relies upon ... "induction" in the manner followed in natural science." John Wallis reliance on induction raised criticism from many other mathematicians, including Fermat. Wallis answer these criticisms in *A treatise of algebra*, p. 306:

> As to the thing itself, I look upon induction as a very good Method of Investigation; as that which doth very often lead us to the easy discovery of a General Rule; or is at least a good preparative to such an one. And where the Result of such Inquiry afforts to the view, an obvious discovery; it needs not (though it may be capable of it,) any further Demonstration. And so it is, when we find the Result of such Inquiry, to put us into a regular progression (of what nature soever,) which is observable to proceed according to one and the same general Process; and where there is no ground of suspicion why it should fail, or of any case which might happen to alter the course of such Process.
>
> Such observation would be looked upon, as sufficiently instructive; since there is no reason of Suspicion, why it should not so continually proceed ...
>
> ... such induction hath been hitherto thought (by such as do not list to be captious) a conclusive Argument.
>
> And the same may be said of all the Inductions which I make use of; Which I always pursue so far (by regular demonstration, where it is no so obvious as not to need it,) till it lead me into a regular or derly Process; and for the most part (if not always) to an Arithmetical Procession; in which I acquiefs as a sufficient evidence, when there is no colour of pretence why it should be thought not to proceed onward in like manner.

[Related or Clarifying References]

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- Charles Babbage, *Passages from the Life of a Philosopher*, 1864.
- Dugald Stewart, *The collected works of Dugald Stewart*, Vol 3, 1855, pp. 396–7.
- Dugald Stewart, ”Supplemental Observations on the Words Induction and Analogy, as used in Mathematics”, *The collected works of Dugald Stewart*, Vol 3, 1855, pp. 316–322.
- Lenore Feigenbaum, ”Brook Taylor and the method of increments”, *Archive for History of Exact Sciences*, 4.X.1985, Volume 34, Issue 1–2, pp. 1–140.
- Giovanni Ferraro, ”The Bernoulli series and Leibniz's analogy”, *The Rise and Development of the Theory of Series up to the Early 1820s (Sources and Studies in the History of Mathematics and Physical Sciences)*, 2008, pp. 45–51.
- Florian Cajori, ”Origin of the Name ”Mathematical Induction””, *The American Mathematical Monthly*, 25(5), 1918.
- John Wallis, ”Of Mons. Fermat's Exceptions to it”, *A Treatise of Algebra*, Chapter LXXIX, 1685, p. 305-9.

]

[Introduction to *Essays on the Philosophy of Analysis*]

In presenting to the Cambridge Philosophical Society the first of a series of *Essays on the Philosophy of Analysis*[,] some account of the object and nature of the dissensions they will contain as well as a sketch of the path I propose to pursue may render their intimate connection more evident and may remove that appearance of disconnection which an attention to their titles alone is in some degree calculated to suggest.

It is my intention[,] in the following pages[,] to attempt an examination of some of those modes by which mathematical discoveries have been made[,] to point out some of those evanescent links which but rarely appear in the writings of the discoverer[,] but which passing perhaps imperceptibly through his mind, have acted as his unerring although his unknown guides.

I would however[,] to avoid misconception[,] state at the outset that I have not attempted to explain the nature of the inventive faculty[,] nor am I of the opinion that its absence could be supplied by rules however skilfully contrived[.] [A]ll that I have proposed is[,] by an attentive examination of the writing of those who have contributed most to the advancement of mathematical science and by a continued attention to the operations of my own mind[,] to state in words some of those principles which appear to me to exercise a very material influence in directing the intellect in its transition from the known to the unknown.

I do not imagine that the enumeration I have given of those modes of operation is complete[.] [S]everal other may probably be added exercising an influence perhaps as powerful as those which I have discussed[.]

I have been more anxious to establish those which I have proposed as firm foundations and to render them useful by numerous illustrations than vainly to attempt to exhaust them and to present a number of barren rules whose justice would only be acknowledged by an examination of the examples which ought to support them and whose utility would be extremely limited from the absence of such accomplishments.

Of the utility of such an undertaking[,] I can hardly persuade myself that much need be said[,] and[,] whatever may be its success[,] it can scarcely fail to excite the attention of those who posses the rare talent of invention[,] to the operations of their mind during its exercise[,] and may perhaps induce some future enquirer more skilfully to arrange the valuable material which such observations could not fail to provide.

The metaphysics of abstract science have hitherto been productive of little which has added to its extent[,] although of much which has elucidated its elements and examined the perspicuity and elegance of its methods[.]

I can however far from agreeing with the opinion expressed by a celebrated author who has successfully applied himself to this department of science[,] M. Carnot -

> The metaphysics of science cannot contribute much to the advancement of methods but there are people who make it a favorite study, and it is for them that I made this booklet.[[57]]

As far as this relates solely to a sound explanation of their rationale of methods or an examinations of their first principles[,] I am inclined to concur with him[,] but I am much deceived by the following essays[,] imperfect as they undoubtedly are[,] will not go far to refute it in any more extensive signification. Indeed so strongly have I always been impressed with the utility of such enquiries to the progress of mathematical science[,] that the [later?] research has constantly been solaced by the additional satisfaction which arose from considering that their successful termination would display in a strong light the advantages which result from a proper application of the philosophy of mind to other sciences[,] advantages which has not been sufficiently appreciated[,] rather from the want of examples in the application than from any defect in the powers of the instruments[.]

The titles of the several essays which will contain the result of my enquiries are as follows[:] 1st Of the influence of signs in mathematical reasoning[.] 2nd Of Notation. 3rd General notions respecting analysis. These three are intended as an introduction which will explain some difficulties and introduce greater uniformity into mathematical symbols. Those which follow relate more immediately to the subject of invention and are 4th Of induction, 5th Of Generalization [,]6[th] Of Analogy. 7th Of the law of Continuity[.] 8[th] Of the use of a register of ideas which occasionally strike the mind. 9[th] Of Artifices, 10[th] I scarcely know what name I shall attach to the tenth essay as the want of an [E]nglish one has hitherto compelled me to employ a very significant foreign term and to entitle it Des rapprochements. The 11[th] and last essay which is of some interest consists of a variety of problems requiring the invention of new modes of analysis. Such is the plan which I have proposed the arrangement may perhaps will maybe a little varied by the substance scarcely altered.

In thus exploring of new and difficult path[,] I do not flatter myself with the idea that all the principles I shall have to state will be expressed in that form which shall ultimately be most fitted to them[,] nor even that some of them may not be exposed to objections I may be unable to obviate[.] [I]n such cases[,] I shall not be backward to modifying or [rejecting?] them according to the weight of the reasons which may be

[57] From Lazare Carnot, *Reflections on the Metaphysics of the Infinitesimal Calculus Réflections sur la métaphysique du calcul infinitésimal*, Paris, 1797, p. 218–9. Translated by Franck Chauvel from "La métaphysique des sciences peut ne pas contribuer beaucoup au progrès des méthodes mais il y a des personnes qui s'en font une étude favorite, et c'est pour eux que j'ai composée cet opuscule." from *Réflections sur la métaphysique du calcul infinitésimal*

produced in the discussion. On the perusals of these pages there will doubtless occur to many of the members of this society illustrations which have altogether escaped their own observation[,] and which may either confirm or refute the doctrines I have supported. The communication of these or even the most brief [response?] to them would considerably facilitate any further researches on the subject[,] and[,] to whatever first they may tend[,] I shall feel equally indebted to those who may favour me with [attention?][.]